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EXPONENTIAL STABILITY OF SOLUTIONS FOR A SYSTEM OF VISCOELASTIC WAVE EQUATIONS WITH VARIABLE COEFFICIENTS; PAST HISTORY AND LOGARITHMIC NONLINEARITIES

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Abstract. This paper deals with coupled wave system with viscoelastic terms and new logarithmic nonlinearities. We use the theory of semigroup to establish a theorem of local existence. Under some specific conditions, we prove that the solution is global in time. After that, we give the exponential stability result of the global solution. The introduced logarithmic nonlinearities are partial derivatives of a primitive function **F(u,v)**, this function is not necessarily a positive function, which creates obstacles and problems to get the existence and the stability result. Our goal in this work is to overcome this challenge and give new solutions to analyze this type of nonlinear systems. Our new result provides a step forward in how to deal with coupled wave systems.

Keywords: Exponential stability, global existence, logarithmic nonlinearity, viscoelastic, wave equation.
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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^2$, with a smooth boundary $\partial\Omega$. In this paper, we consider the coupled system of viscous wave equations with variable coefficients:

$$\begin{cases} u_{tt} + Lu - \displaystyle\int_0^\infty g(s)\, Lu(t-s)\, ds = \kappa v^{2p} u^{2p-1} \ln(|uv|), & \text{in } \Omega \times (0,\infty), \\ v_{tt} + Lv - \displaystyle\int_0^\infty g(s)\, Lv(t-s)\, ds = \kappa v^{2p-1} u^{2p} \ln(|uv|), & \text{in } \Omega \times (0,\infty), \\ u = v = 0, & \text{on } \Gamma \times (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), & \text{in } \Omega, \end{cases} \tag{1}$$

with $Lu = -div(A\nabla u) = -\sum\limits_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$, $p \geq 2$ and $\kappa \in (0, \varrho_0)$ where $\varrho_0 > 0$. u_0, u_1, v_0 and v_1 are given initial data.The function g denotes the kernels of memory terms. The logarithmic nonlinearity appears naturally in inflation cosmology and supersymmetric field theories, quantum mechanics, and many other branches of physics such as nuclear physics, optics, and geophysics, for more applications see Bartkowski & Gorka (2008); Białynicki-Birula & Mycielski (1975); Górka (2009); Barrow & Parsons (1995); Enqvist & McDonald (1998). These specific

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applications in physics and other fields attracted a lot of mathematical researchers to work with such problems. Cazenave & Haraux (1980) established the existence and uniqueness of a solution to the Cauchy problem for the equation

$$u_{tt} - \Delta u = u \log|u|^k, \tag{2}$$

in $\mathbb{R}^3$ and k a positive constant. Using compactness method, Górka (2009) established the global existence of weak solutions for all $(u_0, u_1) \in H_0^1 \times L^2$ to the initial boundary value problem of equation (2) in the one-dimensional case. Bartkowski & Gorka (2008), obtained the existence of classical solutions and investigated weak solutions for the corresponding Cauchy problem of equation (2) in the one-dimensional case. Han (2013), studied the global existence of weak solutions for the initial boundary value problem

$$\begin{cases} u_{tt} - \Delta u + u - u\log|u|^2 + u_t + u|u|^2 = 0, & \text{in } \Omega \times (0,T), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{3}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^3$. Hiramatsu et al. (2010) gave a numerical study of the model (3). However, there is no theoretical analysis for the problem as in Han (2013). In addition, Peyravi (2020) improved and extended some previous studies such as the one by Hu et al. (2019). He studied the decay estimate and exponential growth of solutions for the problem

$$\begin{cases} u_{tt} - \Delta u + u + \displaystyle\int_0^t g(t-s)\Delta u(s)ds + h(u_t)u_t \ + |u|^2 u = u\ln|u|^k, & \text{in } \Omega \times (0,\, T), \\ u(x,\, t) = 0, & x \in \partial\Omega,\ t \in (0,\, T), \\ u(x,\, 0) = \varphi(x),\ u_t(x,\, 0) = \psi(x), & x \in \Omega, \end{cases} \tag{4}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega, h(s) = k_0 + k_1|s|^{m-1}$ where k, k_0, k_1 and m are positive constants, g represents the memory kernel and satisfying

$$g(0) > 0, \quad \int_0^{+\infty} g(s)ds < +\infty, \quad 1 - \int_0^{+\infty} g(s)ds = l_0 > 0. \tag{5}$$

Al-Gharabli et al. (2019) considered the following plate equation:

$$\begin{cases} u_{tt} + \Delta^2 u + u - \displaystyle\int_0^t g(t-s)\Delta^2 u(s)ds = ku\ln|u|, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{6}$$

where Ω is a bounded domain of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$ and k is a small positive real number. They proved the existence and decay results of the solutions, imposing the condition on the relaxation function:

$$g'(t) \le -\xi(t)g^p(t), \quad 1 \le p < 3/2. \tag{7}$$

Regarding the system of wave equations without logarithmic source term the initial boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds + h(u_t) = f_1(u,v), & \text{in } \Omega \times (0,\infty), \\ v_{tt} - \Delta v + \displaystyle\int_0^t g_2(t-s)\Delta v(s)ds + h(v_t) = f_2(u,v), & \text{in } \Omega \times (0,\infty) \\ u = v = 0, & \text{on } \partial\Omega \times [0,\infty), \\ u(\cdot,0) = u_0,\ u_t(\cdot,0) = u_1,\ v(\cdot,0) = v_0,\ v_t(\cdot,0) = v_1, & \text{in } \Omega, \end{cases} \tag{8}$$

has been considered, where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The systems of wave equation have been extensively studied and several results have been obtained. See, in this regard, previous studies, e.g. Said-Houari et al. (2011); Han & Wang (2009); Mustafa (2012); Messaoudi & Al-Gharabli (2015). Concerning the viscoelastic systems with logarithmic non-linearities, there are numerous results related to the asymptotic behavior of solutions. For example, Wang et al. (2019) studied the problem

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta u + u_t = |u|^{k-2}u\ln|u|, & \text{in } \Omega\times(0,\infty),\\ v_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta v + v_t = |v|^{k-2}v\ln|v|, & \text{in } \Omega\times(0,\infty),\\ u(.,0)=u_0,\ u_t(.,0)=u_1, & \text{in } \Omega,\\ v(.,0)=v_0,\ v_t(.,0)=v_1, & \text{in } \Omega,\\ u=v=0, & \text{in } \partial\Omega\times(0,\infty), \end{cases} \tag{9}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^n$. The term $M(s) := \alpha + \beta s^\gamma$ is a Kirchhoff term, where $\alpha \geq 1, \beta \geq 0, \gamma > 0$ and $k \geq 2\gamma + 2$. By employing potential well method and concavity method, they obtained several results related to the sufficient conditions posed on subcritical initial energy and critical initial energy, which is used to classify initial data for global existence and finite time blow up. Boulaaras (2021) studied the coupled Lamé system

$$\begin{cases} u_{tt} + \alpha v - \Delta_e u + \displaystyle\int_0^t g_1(t-s)\Delta u(s)\mathrm{d}s - \mu_1\Delta u_t(t) = b_1 u\ln|u(t)|, & \text{in } \Omega\times(0,+\infty),\\ v_{tt} + \alpha u - \Delta_e v + \displaystyle\int_0^t g_2(t-s)\Delta v(s)\mathrm{d}s - \mu_2\Delta v_t(t) = b_2 v\ln|v(t)|, & \text{in } \Omega\times(0,+\infty),\\ u(.,0)=u_0,\ u_t(.,0)=u_1, & \text{in } \Omega,\\ v(.,0)=v_0,\ v_t(.,0)=v_1, & \text{in } \Omega,\\ u=v=0, & \text{in } \partial\Omega\times(0,\infty), \end{cases} \tag{10}$$

where Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega, \mu_1, \mu_2, \alpha, b_1, b_2$ are positive constants and (u_0, u_1, v_0, v_1) are given as history and initial data. Here Δ_e refers to the elasticity operator, which is defined as

$$\Delta_e u = \mu\Delta u + (\lambda+\mu)\nabla(\operatorname{div} u),\ u = (u_1, u_2, u_3)^{\mathrm{T}}.$$

He obtained exponential decay of solutions, imposing the condition on the relaxation function,

$$g_i'(t) \leq -\xi(t)g_i(t), \qquad \text{for all } t \geq 0 \qquad \text{for } i = 1, 2$$

where $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function. Very recently, Irkıl et al. (2022) investigated the problem

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta u + \displaystyle\int_0^t g_1(t-s)\Delta u(s)ds - \Delta u_t = |v|^{p-2}u\ln|u|, & \text{in } \Omega\times(0,\infty),\\ v_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta v + \displaystyle\int_0^t g_2(t-s)\Delta v(s)ds - \Delta v_t = |v|^{p-2}v\ln|v|, & \text{in } \Omega\times(0,\infty),\\ u(.,0)=u_0,\ u_t(.,0)=u_1, & \text{in } \Omega,\\ v(.,0)=v_0,\ v_t(.,0)=v_1, & \text{in } \Omega,\\ u=v=0, & \text{in } \partial\Omega\times(0,\infty), \end{cases} \tag{11}$$

where $p \geq 2\gamma + 2$ is a real number. $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a regular and bounded domain with smooth boundary $\partial\Omega$. Here, M is a positive C^1 function for $s \geq 0$ satisfying $M(s) = \beta_1 + \beta_2 s^\gamma, \gamma > 0, \beta_1 \geq 1, \beta_2 \geq 0$. The kernels $g_i : R^+ \to R^+ (i = 1, 2)$ satisfying

$$g_i'(t) \leq -\varrho g_i(t), \qquad t \geq 0, \tag{12}$$

for some positive constant ϱ. The authors proved the global existence and they established decay rate estimates by using multiplier method.

Challenge

It is well known that f_1 and f_2 in (8) have a relation (see for example Said-Houari et al. (2011); Han & Wang (2009); Mustafa (2012); Messaoudi & Al-Gharabli (2015)), which is the existence of a **positive** function $F(u,v)$ satisfying

$$\begin{cases} f_1(u,v) = \dfrac{\partial F}{\partial u}(u,v), \\ \\ f_2(u,v) = \dfrac{\partial F}{\partial v}(u,v), \\ \\ \mathbf{F(u,v) \geq 0}. \end{cases} \tag{13}$$

But in our system $f_1(u,v) = \frac{\partial F}{\partial u}(u,v)$ and $f_2(u,v) = \frac{\partial F}{\partial v}(u,v)$ where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by

$$F(u,v) = \frac{\kappa}{4p^2}\left(v^{2p}u^{2p}\ln\left(|uv|^{2p}\right) - v^{2p}u^{2p}\right),$$

is not necessarily a positive function. In this paper, we are concerned with coupled system (1) with variable coefficients and **new** logarithmic non-linearity terms. We introduce these non-linear terms to generalize the previous publications and to highlight how to deal with the primitive function $F(u,v)$. We prove that the solution is global in time when κ belongs to a specific interval from the set of real positive numbers.

In the following section, we will go through several notations and assumptions that we will require for our work. The third section is devoted to prove the existence of the local and global solution. In the forth section we show several technical lemmas that are required for our main result. We state and prove our stability result in the last section.

2 Preliminary

In this section, we present some material that we shall use in order to present our results. Denote $V = H_0^1(\Omega)$ and $(u,v) = \int_\Omega u(x,t)v(x,t)\mathrm{d}x$ the scalar product in $L^2(\Omega)$. Also we mean by $\|.\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$. The Poincaré inequality holds on V , i.e. there exits a constant C_* such that

$$\forall u \in V, \ \ \|u(t)\|_q \ \leq C_*\|\nabla u(t)\|_2. \tag{14}$$

For studying the problem (1), we will need the assumptions:

$(A1)$ $g : [0, \ \infty) \to (0, \ \infty)$ is C^2 nonincreasing differentiable function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = \ell > 0.$$

$(A2)$ There exists a positive nonincreasing differentiable function $\xi(t)$ such that

$$g'(t) \leq -\xi(t)g(t), \qquad \forall t \geq 0, \tag{15}$$

where ξ satisfies

$$\int_0^{+\infty} \xi(t)dt = +\infty. \tag{16}$$

$(A3)$ The matrix $A(x) = (a_{ij}(x))$, with $a_{ij} \in C^1\left(\overline{\Omega}\right)$, is symmetric and there exists a constant $a_0 > 0$ such that for all $x \in \Omega$ and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$, we have

$$\sum_{i,j=1}^{2} a_{ij}(x)\zeta_j\zeta_i \geq a_0|\zeta|^2. \tag{17}$$

Following the same arguments of Dafermos (1970), we introduce new variables to establish the usual history setting of problem (1):

$$\vartheta^t(x,s) = u(x,t) - u(x,t-s), \qquad s,\ t \in \mathbb{R}_+ \tag{18}$$

and

$$\varphi^t(x,s) = v(x,t) - v(x,t-s), \qquad s,\ t \in \mathbb{R}_+. \tag{19}$$

Therefore, problem (1) takes the form

$$\begin{cases} u_{tt} + \ell Lu + \displaystyle\int_0^{+\infty} g(s)\, L\vartheta^t(s)\, ds = \kappa v^{2p} u^{2p-1} \ln(|uv|), & \text{in } \Omega \times (0,\infty), \\ v_{tt} + \ell Lv + \displaystyle\int_0^{+\infty} g(s)\, L\varphi^t(s)\, ds = \kappa v^{2p-1} u^{2p} \ln(|uv|), & \text{in } \Omega \times (0,\infty), \end{cases} \tag{20}$$

and

$$\begin{cases} \vartheta_t^t(x,t) + \vartheta_s^t(x,s) = u_t(x,t), & x \in \Omega,\ s,\ t \in (0,\infty), \\ \varphi_t^t(x,t) + \varphi_s^t(x,s) = v_t(x,t), & x \in \Omega,\ s,\ t \in (0,\infty), \\ u(x,t) = v(x,t) = \vartheta^t(x,s) = \varphi^t(x,s) = 0, & x \in \Gamma,\ s,\ t \in (0,\infty), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), & x \in \Omega, \\ \vartheta^0(x,s) = \vartheta_0(x,s) = u(x,0) - u(x,-s), & x \in \Omega,\ s \in (0,\infty), \\ \varphi^0(x,s) = \varphi_0(x,s) = w(x,0) - w(x,-s) & x \in \Omega,\ s \in (0,\infty). \end{cases} \tag{21}$$

Lemma 1. *There exists $\tau > 0$ such that*

$$y|\ln y| \le y^2 + \tau\sqrt{y}, \quad \forall y > 0. \tag{22}$$

We follow the same technique as Al-Gharabli et al. (2020) to prove this lemma.

Proof. Let $r(y) = \sqrt{y}(|\ln y| - y)$. Notice that r is continuous on $(0,\infty)$ and its limit at $0^+$is 0^+, and its limit at $+\infty$ is $-\infty$. Then r has a maximum τ on $(0,\infty)$, so (22) holds. □

Lemma 2. *There exists a positive constant $A > 0$, such that the real function z defined by*

$$z(y) = \begin{cases} y^2 \ln|y|, & y \ne 0, \\ 0, & y = 0, \end{cases}$$

satisfies

$$|z(y)| \le |y|^3 + A, \text{ for all } y \in \mathbb{R}. \tag{23}$$

Proof. As in Kafini & Messaoudi (2020), we have $\lim\limits_{|y|\to+\infty} \left(\dfrac{\ln|y|}{|y|}\right) = 0$, then there exists $C > 0$ such that

$$\frac{\ln|y|}{|y|} < 1, \quad \forall\ |y| > C$$

Then,

$$|z(y)| \le |y|^3, \quad \forall\ |y| > C.$$

Since $\lim\limits_{|y|\to 0} z(y) = 0$, then $|z(y)| \le A$, for some $A > 0$ and for all $|y| \le C$. Thus,

$$|z(y)| \le |y|^3 + A.$$

□

Lemma 3. *The following inequality holds for any $y > 0$,*

$$y^2 \ln y \le y^3. \tag{24}$$

Proof. We have $\dfrac{\ln(y)}{y} \le 1$, then by multiplication this inequality by y^3, we get (24). □

3 Well-Posedness

Throughout this paper, c is used to denote a positive generic constant. Let us introduce the notations

$$\left(g \diamond \nabla\vartheta^t\right)(t) = \int_0^\infty g(s) a\left(\vartheta^t(s), \vartheta^t(s)\right) \mathrm{d}s,$$

where

$$a(w(t), z(t)) = \sum_{i,j=1}^{2} \int_\Omega a_{ij}(x) \frac{\partial w(t)}{\partial x_j} \frac{\partial z(t)}{\partial x_i} dx = \int_\Omega A \nabla w(t) \nabla z(t) dx \tag{25}$$

Remark 1. By using (A3), we verify that the bilinear form $a(.,.) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is symmetric and continuous. In addition, from (17), we have $a(u(t), u(t)) \geq a_0 \int_\Omega \sum_{i=1} |\frac{\partial u}{\partial x_i}|^2 dx = a_0 \|\nabla u(t)\|_2^2$, which implies that $a(.,.)$ is coercive.

Now, we introduce the energy associated to problem (20)

$$\begin{aligned} E(t) =& \frac{1}{2}\|u_t(t)\|_2^2 + \frac{1}{2}\|v_t(t)\|_2^2 + \frac{\ell}{2} a(u(t), u(t)) + \frac{\ell}{2} a(v(t), v(t)) \\ &+ \frac{1}{2}\left(g \diamond \nabla\vartheta^t\right)(t) + \frac{1}{2}\left(g \diamond \nabla\varphi^t\right)(t) - \kappa \int_\Omega F(u, v) dx, \ \forall t \geq 0, \end{aligned} \tag{26}$$

where $F(u, v) = \frac{1}{4p^2}\left(v^{2p} u^{2p} \ln\left(|uv|^{2p}\right) - v^{2p} u^{2p}\right)$.

Lemma 4. *Let (u, v) be the solution of (20). Then, the energy functional defined by (26) is nonincreasing and we have for all $t \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) = \frac{1}{2}\left(g' \diamond \nabla\varphi^t\right)(t) + \frac{1}{2}\left(g' \diamond \nabla\vartheta^t\right)(t). \tag{27}$$

Proof. Multiplying the first equation in (20) by u_t and the second by v_t, integrating over Ω, using Green's formula and exploiting the forth and the fifth equations in system (20), we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E(t) =& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(g \diamond \nabla\vartheta^t\right)(t) - \int_\Omega \int_0^\infty g(s) A \nabla\vartheta^t(s) \nabla u_t(t) \mathrm{d}s\mathrm{d}x \\ &+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(g \diamond \nabla\varphi^t\right)(t) - \int_\Omega \int_0^\infty g(s) A \nabla\varphi^t(s) \nabla v_t(t) \mathrm{d}s\mathrm{d}x. \end{aligned} \tag{28}$$

Using the fact that $\vartheta_t^t(s) + \vartheta_s^t(s) = u_t(t)$, the fourth term in the right-hand side of (27) can be written as

$$\begin{aligned} &- \int_\Omega \int_0^\infty g(s) A \nabla\vartheta^t(s) \nabla u_t(t) \mathrm{d}s\mathrm{d}x = \\ &- \int_\Omega \int_0^\infty g(s) A \nabla\vartheta^t(s) \nabla\vartheta_s^t(s) \mathrm{d}s\mathrm{d}x - \int_\Omega \int_0^\infty g(s) A \nabla\vartheta^t(s) \nabla\vartheta_t^t(s) \mathrm{d}s\mathrm{d}x \\ =& \frac{1}{2} \int_\Omega \int_0^\infty g'(s) A \nabla\vartheta^t(s) \nabla\vartheta^t(s) \mathrm{d}s\mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(g \diamond \nabla\vartheta^t\right)(t). \end{aligned} \tag{29}$$

In the same way, we get

$$\begin{aligned} &- \int_\Omega \int_0^\infty g(s) A \nabla\varphi^t(s) \nabla v_t(t) \mathrm{d}s\mathrm{d}x = \\ &\frac{1}{2} \int_\Omega \int_0^\infty g'(s) A \nabla\varphi^t(s) \nabla\varphi^t(s) \mathrm{d}s\mathrm{d}x - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(g_2 \diamond \nabla\varphi^t\right)(t). \end{aligned} \tag{30}$$

Using (28), (29) and (30), we obtain desired result. □

Now, we start to give the Well-Posedness of our problem. $L_g^2(\mathbb{R}_+;V)$ denotes the Hilbert space of V -valued functions on $\mathbb{R}_+$ endowed with the inner product

$$\langle \chi, \psi\rangle_{L_g^2(\mathbb{R}_+;V)} = \int_\Omega \left(\int_0^\infty g(s)A(x)\nabla\chi(s)\nabla\psi(s)ds \right) dx. \tag{31}$$

Denote by $\mathbf{H}$ the Hilbert space

$$\mathbf{H} = V^2 \times \left(L^2(\Omega)\right)^2 \times \left(L_g^2\left(\mathbb{R}_+;V\right)\right)^2$$

equipped with the inner product

$$\begin{aligned}
&\left\langle (u,w,\psi,\chi,\vartheta,\theta)^T, (\widehat{u},\widehat{w},\widehat{\psi},\widehat{\chi},\widehat{\vartheta},\widehat{\theta})^T \right\rangle_{\mathbf{H}} \\
&= \ell\int_\Omega A(x)\nabla u\nabla\widehat{u}dx + \ell\int_\Omega A(x)\nabla w\nabla\widehat{w}dx + \int_\Omega \psi\widehat{\psi}dx + \int_\Omega \chi\widehat{\chi}dx \\
&+ \int_\Omega\int_0^\infty g(s)A(x)\nabla\vartheta\nabla\widehat{\vartheta}dsdx + \int_\Omega\int_0^\infty g(s)A(x)\nabla\theta\nabla\widehat{\theta}dsdx.
\end{aligned}$$

Let us denote $U = \left(u, v, u_t, v_t, \vartheta^t, \varphi^t\right)^T$. The problem (20) can be rewritten as

$$\begin{aligned}
&U' = \mathscr{A}U + \mathcal{J}(U), \qquad t > 0 \\
&U(0) = (u_0, v_0, u_1, w_1, \vartheta_0, \varphi_0)^T,
\end{aligned} \tag{32}$$

where the operator $\mathscr{A}$ is defined by

$$\mathscr{A}\begin{pmatrix} u \\ v \\ \psi \\ \chi \\ \vartheta \\ \theta \end{pmatrix} = \begin{pmatrix} \psi \\ \chi \\ -\ell Lu - \displaystyle\int_0^{+\infty} g(s)\, L\vartheta(s)\, ds \\ -\ell Lv - \displaystyle\int_0^{+\infty} g(s)\, L\theta(s)\, ds \\ -\vartheta_s + \psi \\ -\theta_s + \chi \end{pmatrix} \tag{33}$$

and $\mathcal{J}(U) = (0, 0, \kappa v^{2p}u^{2p-1}\ln(|uv|), \kappa v^{2p-1}u^{2p}\ln(|uv|), 0, 0) = (0, 0, f_1(u,v), f_2(u,v), 0, 0)$, with domains

$$D(\mathscr{A}) = \left\{ \begin{array}{l} (u,w,\psi,\chi,\vartheta,\theta)^T \in \mathbf{H} : (u,v) \in \left(H^2(\Omega)\cap V\right)^2,\ (\psi,\chi) \in V, \\ (\vartheta,\theta) \in \left(L_g^2\left(\mathbb{R}_+; H^2(\Omega)\cap V\right)\right)^2,\ (\vartheta_s,\theta_s) \in \left(L_g^2\left(\mathbb{R}_+;V\right)\right)^2, \\ \vartheta(0) = \theta(0) = 0, \end{array} \right\}$$

and $D(\mathcal{J}) = \mathbf{H}$. First, we prove that the operator $\mathcal{J} : \mathbf{H} \to \mathbf{H}$ is locally Lipschitz. We see that

$$\left\| \mathcal{J}(U) - \mathcal{J}(\tilde{U}) \right\|_{\mathbf{H}}^2 = \|f_1(u,v) - f_1(\tilde{u},\tilde{v})\|_2^2 + \|f_2(u,v) - f_2(\tilde{u},\tilde{v})\|_2^2.$$

Let us define the $C^1(\mathbb{R})$ function by

$$h(s) = \begin{cases} s^{2p-1}\ln(|s|),\ s \neq 0, \\ 0,\ s = 0 \end{cases} \quad \text{and } h'(s) = \begin{cases} (2p-1)\, s^{2p-2}\ln(|s|) \\ +s^{2p-2},\ s \neq 0, \\ 0, \quad s = 0. \end{cases} \tag{34}$$

Hence,

$$\begin{aligned}
|f_1(u,v) - f_1(\bar{u},\bar{v})| &= \kappa|v^{2p}u^{2p-1}\ln(|uv|) - \bar{v}^{2p}\bar{u}^{2p-1}\ln(|\bar{u}\bar{v}|)| \\
&= \kappa|vh(uv) - vh(\bar{u}\bar{v}) + vh(\bar{u}\bar{v}) - \bar{v}h(\bar{u}\bar{v})| \\
&\leq \kappa|v||h(uv) - h(\bar{u}\bar{v})| + \kappa|h(\bar{u}\bar{v})||v - \bar{v}|
\end{aligned} \tag{35}$$

and

$$
\begin{aligned}
|f_2(u,v)-f_2(\bar{u},\bar{v})| =& \kappa|u^{2p}v^{2p-1}\ln(|uv|)-\bar{u}^{2p}\bar{v}^{2p-1}\ln(|\bar{u}\bar{v}|)| \\
=& \kappa|uh(uv)-uh(\bar{u}\bar{v})+uh(\bar{u}\bar{v})-\bar{u}h(\bar{u}\bar{v})| \\
\leq& \kappa|u||h(uv)-h(\bar{u}\bar{v})|+\kappa|h(\bar{u}\bar{v})||u-\bar{u}|.
\end{aligned}
\tag{36}
$$

As a consequence of the mean value theorem, we have, for $0 \leq \lambda \leq 1$,

$$
\begin{aligned}
I_3 &= |h(uv)-h(\bar{u}\bar{v})| \\
&= |h'(\lambda uv+(1-\lambda)\bar{u}\bar{v})||uv-\bar{u}\bar{v}| \\
&= |\frac{(2p-1)}{p-1}(\lambda uv+(1-\lambda)\bar{u}\bar{v})^{2(p-1)}\ln\left(|\lambda uv+(1-\lambda)\bar{u}\bar{v}|^{p-1}\right)+ \\
&(\lambda uv+(1-\lambda)\bar{u}\bar{v})^{2p-2}|\times|uv-v\bar{u}+v\bar{u}-\bar{u}\bar{v}|.
\end{aligned}
\tag{37}
$$

Using lemma 2, Young's inequality and exploiting $(x+y)^r \leq 2^{r-1}(x^r+y^r), x,y>0, r>1$, we get

$$
\begin{aligned}
I_3 \leq& \left[\frac{(2p-1)}{p-1}\left(|\lambda uv+(1-\lambda)\bar{u}\bar{v}|^{3(p-1)}+A\right)+(2)^{-1}(2\lambda)^{2p-2}|uv|^{2p-2}\right. \\
&\left.+(2)^{-1}(2(1-\lambda))^{2p-2}|\bar{u}\bar{v}|^{2p-2}\right]\times[|v||u-\bar{u}|+|\bar{u}||v-\bar{v}|] \\
\leq& c\left[c+|uv|^{3(p-1)}+|\bar{u}\bar{v}|^{3(p-1)}+|uv|^{2p-2}+|\bar{u}\bar{v}|^{2p-2}\right]\times \\
&[|v||u-\bar{u}|+|\bar{u}||v-\bar{v}|] \\
\leq& c\left[c+|v|^2+|u|^{6(p-1)}+|v|^{6(p-1)}+|\bar{u}|^{6(p-1)}\right. \\
&\left.+|\bar{v}|^{6(p-1)}+|u|^{4(p-1)}+|v|^{4(p-1)}+|\bar{u}|^{4(p-1)}+|\bar{v}|^{4(p-1)}\right]|u-\bar{u}| \\
&+c\left[c+|\bar{u}|^2+|u|^{6(p-1)}+|v|^{6(p-1)}+|\bar{u}|^{6(p-1)}+|\bar{v}|^{6(p-1)}\right. \\
&\left.+|u|^{4(p-1)}+|v|^{4(p-1)}+|\bar{u}|^{4(p-1)}+|\bar{v}|^{4(p-1)}\right]|v-\bar{v}|.
\end{aligned}
\tag{38}
$$

Now, we use lemma 2 to estimate $|h(\bar{u}\bar{v})|$ as

$$
\begin{aligned}
|h(\bar{u}\bar{v})| =& \frac{1}{p-1}|(\bar{u}\bar{v})(\bar{u}\bar{v})^{2(p-1)}\ln(|\bar{u}\bar{v}|^{p-1})| \\
\leq& |\bar{u}\bar{v}|^{3p-1}+A|\bar{u}\bar{v}| \\
\leq& c\left[|\bar{u}|^{6(p-1)}+|\bar{v}|^{6(p-1)}+|\bar{u}|^2+|\bar{v}|^2\right].
\end{aligned}
\tag{39}
$$

Finally, we combine (35)-(39) and using Hölder's inequality with Sobolev embedding, we arrive at

$$
\begin{aligned}
&\|f_1(u,v)-f_1(\bar{u},\bar{v})\|_2^2+\|f_2(u,v)-f_2(\bar{u},\bar{v})\|_2^2 \\
&\leq C\left(\|u\|_V,\|v\|_V,\|\bar{u}\|_V,\|\bar{v}\|_V\right)\left[\|u-\bar{u}\|_V^2+\|v-\bar{v}\|_V^2\right].
\end{aligned}
\tag{40}
$$

Therefore, $\mathcal{J}$ is locally Lipschitz.
By using the semi-group approach and combining the ideas from Guesmia (2011), we can prove that the operator $\mathscr{A}$ generates a monotone maximal operator on H and therefore the next existence result holds (see Pazy (1983),Komornik (1994)).

Theorem 1. *Assume that (A1)-(A3) hold. Then for any $U_0 \in \boldsymbol{H}$, there exists a unique solution $U \in C([0,T];\boldsymbol{H})$ of problem (32).*

3.1 Global existence

First, we introduce the functionals:

$$\begin{aligned} J(t) = & \frac{\ell}{2} \times a(u(t), u(t)) + \frac{\ell}{2} \times a(v(t), v(t)) + \frac{1}{2}\left(g \diamond \nabla \vartheta^t\right)(t) + \frac{1}{2}\left(g \diamond \nabla \varphi^t\right)(t) \\ & - \kappa \times \int_\Omega F(u, v) \mathrm{d}x. \end{aligned} \tag{41}$$

and

$$\begin{aligned} I(t) = & \ell \times a(u(t), u(t)) + \ell \times a(v(t), v(t)) + \left(g \diamond \nabla \vartheta^t\right)(t) + \left(g \diamond \nabla \varphi^t\right)(t) \\ & - 6\kappa \times \int_\Omega F(u, v) \mathrm{d}x. \end{aligned} \tag{42}$$

In the following sections of the paper, we will assume that

$$\varrho_0 < \min\left\{ \frac{2p^2 \ell^{3p} a_0^{3p}}{3C_*^{6p}}, \frac{p a_0^{3p} \ell^{3p}}{4C_*^{6p}} \right\} = \frac{p a_0^{3p} \ell^{3p}}{4C_*^{6p}}. \tag{43}$$

Lemma 5. *Suppose that (A3) and (43) hold. Then, the inequality*

$$6\kappa \int_\Omega F(u, v) dx \leqslant (\ell a(u, u) + \ell a(v, v))^{3p} \tag{44}$$

holds for any $(u, v) \in V \times V$.

Proof. By using Young's, Poincare's inequalities, (43), lemma 3 and remark 1, we obtain

$$\begin{aligned} \kappa \int_\Omega F(u, v) dx & \leq \kappa \int_\Omega \frac{1}{2p^2} v^{2p} u^{2p} \ln\left(|uv|^p\right) dx \leq \kappa \int_\Omega \frac{1}{2p^2} |uv|^{3p} dx \\ & \leq \kappa \int_\Omega \frac{1}{4p^2} |u|^{6p} dx + \kappa \int_\Omega \frac{1}{4p^2} |v|^{6p} dx \\ & \leq \frac{C_*^{6p}}{4p^2} \kappa \left\|\nabla u\right\|_2^{6p} + \frac{C_*^{6p}}{4p^2} \kappa \left\|\nabla v\right\|_2^{6p} \\ & \leq \kappa \frac{C_*^{6p}}{4p^2 a_0^{3p}} \left(a(u(t), u(t))\right)^{3p} + \kappa \frac{C_*^{6p}}{4p^2 a_0^{3p}} \left(a(v(t), v(t))\right)^{3p} \\ & \leq \frac{\ell^{3p}}{6} \left(a(u(t), u(t))\right)^{3p} + \frac{\ell^{3p}}{6} \left(a(v(t), v(t))\right)^{3p}. \end{aligned} \tag{45}$$

This finishes the proof. $\square$

Lemma 6. *Suppose that* $(A1)-(A3)$ *hold. Then for any* $U_0 \in \boldsymbol{H}$ *satisfying*

$$\left\{ \begin{array}{l} \beta = [3E(0)]^{3p-1} < 1 \\ I(0) = I\left(u_0, v_0\right) > 0 \end{array} \right. \tag{46}$$

we have

$$I(t) = I(u(t), v(t)) > 0, \quad \forall t > 0 \tag{47}$$

Proof. Since $I(0) > 0$, then by continuity,

$$I(t) \geq 0, \quad \text{on } (0, \delta), \delta > 0 \tag{48}$$

Let T_m be such that

$$\left\{I\left(T_m\right) = 0 \quad \text{and} \quad I(t) > 0, \quad \forall 0 \leq t < T_m\right\} \tag{49}$$

which implies that, for all $t \in [0, T_m]$,

$$\begin{aligned} J(t) =& \frac{1}{6} I(t) + \frac{1}{3} \left[\ell a(u(t), u(t)) + \ell a(v(t), v(t)) + \left(g \diamond \nabla \vartheta^t \right)(t) \right. \\ & \left. + \left(g \diamond \nabla \varphi^t \right)(t) \right] \\ \geq & \frac{1}{3} \left[\ell a(u(t), u(t)) + \ell a(v(t), v(t)) + \left(g \diamond \nabla \vartheta^t \right)(t) + \left(g \diamond \nabla \varphi^t \right)(t) \right], \end{aligned} \tag{50}$$

we easily get

$$\begin{aligned} \ell a(u(t), u(t)) + \ell a(v(t), v(t)) & \leq 3J(t) \\ & \leq 3E(t) \leq 3E(0), \quad \forall t \in [0, T_m]. \end{aligned} \tag{51}$$

By exploiting (44), (46) and (51), we obtain

$$\begin{aligned} & 6\kappa \int_\Omega F\left(u\left(T_m\right), v\left(T_m\right)\right) dx \leq (\ell a(u, u) + \ell a(v, v))^{3p} \\ & = \left(\ell a(u\left(T_m\right), u\left(T_m\right)) + \ell a(v\left(T_m\right), v\left(T_m\right))\right)^{3p-1} \\ & \quad \times \left(\ell a(u\left(T_m\right), u\left(T_m\right)) + \ell a(v\left(T_m\right), v\left(T_m\right))\right) \\ & \leq [3E(0)]^{3p-1} \times \left(\ell a(u\left(T_m\right), u\left(T_m\right)) + \ell a(v\left(T_m\right), v\left(T_m\right))\right) \\ & = \beta \left(\ell a(u\left(T_m\right), u\left(T_m\right)) + \ell a(v\left(T_m\right), v\left(T_m\right))\right) \\ & < \left(\ell a(u\left(T_m\right), u\left(T_m\right)) + \ell a(v\left(T_m\right), v\left(T_m\right))\right). \end{aligned} \tag{52}$$

Hence, by using (42), we conclude that

$$I(T_m) > 0. \tag{53}$$

which contradicts our hypothesis (49). So $I(t) > 0$ for all $t \geq 0$,

Theorem 2. *Suppose that $(A1) - (A3)$, and (43) hold. If $U_0 \in \boldsymbol{H}$ satisfying (46), then the solution is global in time.*

Proof. We use (51) to get

$$\begin{aligned} E(0) \geq E(t) = & J(t) + \frac{1}{2} \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) \\ \geq & \frac{1}{3} \left(\ell a(u(t), u(t)) + \ell a(v(t), v(t)) + \left(g \diamond \nabla \vartheta^t \right)(t) + \left(g \diamond \nabla \varphi^t \right)(t) \right) \\ & + \frac{1}{2} \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right). \end{aligned} \tag{54}$$

Therefore,

$$\|U\|_{\mathbf{H}} \leq \bar{c}. \tag{55}$$

That proves the solution is global in time.

□

Remark 2. *The equation (54) ensures that*

$$a(u(t), u(t)) + a(v(t), v(t)) \leq \frac{3}{\ell} E(t). \tag{56}$$

4 Technical lemmas

Before we state our stability theorem and its proof, we establish several lemmas needed to prove our result.

Lemma 7. *Let (A1)-(A3) hold and (u, v) be the solution of (20). We define*

$$K_1(t) := \int_\Omega u u_t dx + \int_\Omega v v_t dx, \tag{57}$$

which satisfies, for all $t \geq 0$,

$$\begin{aligned} K_1'(t) \leq\ & \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \frac{2a_1}{a_0^2}\left(g \diamond \nabla\vartheta^t\right)(t) + \frac{2a_1}{a_0^2}\left(g \diamond \nabla\varphi^t\right)(t) \\ & - \frac{\ell}{2}a(u(t),u(t)) - \frac{\ell}{2}a(v(t),v(t)) \end{aligned} \tag{58}$$

where $a_1 = \max\limits_{1\leq j\leq 2}\sum\limits_{i=1}^{2}\|a_{ij}\|_\infty^2$.

Proof. Taking derivative of $K_1(t)$ and using (20), we get

$$\begin{aligned} K_1'(t) &= \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 - \ell a(u(t),u(t)) - \ell a(v(t),v(t)) \\ &- \int_\Omega\int_0^\infty g(s)A\nabla\vartheta^t(s)\nabla u(t)dsdx - \int_\Omega\int_0^\infty g(s)A\nabla\varphi^t(s)\nabla v(t)dsdx \\ &+ 2\kappa\int_\Omega v^{2p}u^{2p}\ln\left(|uv|\right)dx \end{aligned} \tag{59}$$

Using Young's inequality, (A1) and (A3) the fifth term can be estimated as

$$\begin{aligned} &\int_0^\infty g(s)\int_\Omega A\nabla\vartheta^t(s)\nabla u(t)\mathrm{d}x\ \mathrm{d}s = \sum_{i,j=1}^{2}\int_0^\infty g(s)\int_\Omega a_{ij}(x)\frac{\partial\vartheta^t(s)}{\partial x_j}\frac{\partial u(t)}{\partial x_i}\ \mathrm{d}x\ \mathrm{d}s \\ &\leq \lambda\sum_{i,j=1}^{2}\int_\Omega\left(a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\right)^2\ \mathrm{d}x + \frac{1}{4\lambda}\sum_{i,j=1}^{2}\int_\Omega\left(\int_0^\infty g(s)\frac{\partial\vartheta^t(s)}{\partial x_i}\ \mathrm{d}s\right)^2\ \mathrm{d}x \\ &\leq \frac{\lambda}{a_0}\left(\max_{1\leq j\leq n}\sum_{i=1}^{2}\|a_{ij}\|_\infty^2\right)a(u(t),u(t)) + \frac{2l}{4a_0\lambda}\left(g\diamond\nabla\vartheta^t\right)(t) \\ &\leq \frac{\lambda a_1}{a_0}a(u(t),u(t)) + \frac{\ell}{2a_0\lambda}\left(g\diamond\nabla\vartheta^t\right)(t), \end{aligned} \tag{60}$$

for any $\lambda > 0$. In the same manner, we get

$$\int_\Omega\int_0^\infty g(s)A\nabla\varphi^t(s)\nabla v(t)dsdx \leq \frac{\lambda a_1}{a_0}a(v(t),v(t)) + \frac{\ell}{2a_0\lambda}\left(g\diamond\nabla\varphi^t\right)(t). \tag{61}$$

We can use (45), to estimate the last term as

$$\begin{aligned} &2\frac{\kappa}{p}\int_\Omega v^{2p}u^{2p}\ln\left(|uv|^p\right)dx \\ &\quad\leq \kappa\frac{C_*^{6p}}{pa_0^{3p}}\left(a(u(t),u(t))\right)^{3p} + \kappa\frac{C_*^{6p}}{pa_0^{3p}}\left(a(v(t),v(t))\right)^{3p} \\ &\quad\leq \kappa\frac{(3E(0))^{3p-1}C_*^{6p}}{pa_0^{3p}\ell^{3p-1}}a(u(t),u(t)) + \kappa\frac{(3E(0))^{3p-1}C_*^{6p}}{pa_0^{3p}\ell^{3p-1}}a(v(t),v(t)) \\ &\quad\leq \kappa\frac{C_*^{6p}}{pa_0^{3p}\ell^{3p-1}}a(u(t),u(t)) + \kappa\frac{C_*^{6p}}{pa_0^{3p}\ell^{3p-1}}a(v(t),v(t)) \\ &\quad\leq \frac{\ell}{4}a(u(t),u(t)) + \frac{\ell}{4}a(v(t),v(t)). \end{aligned} \tag{62}$$

Taking $\lambda = \frac{a_0 \ell}{4a_1}$ and combining (59)-(62), we obtain (58). □

Lemma 8. *Under the assumptions (A1)-(A3), the functional K_2 defined by*

$$K_2(t) = R_1(t) + R_2(t) \tag{63}$$

where

$$R_1(t) = -\int_\Omega u_t(t) \int_0^\infty g(s)\vartheta^t(s)\mathrm{d}s \ \mathrm{d}x \tag{64}$$

and

$$R_2(t) = -\int_\Omega v_t(t) \int_0^\infty g(s)\varphi^t(s)\mathrm{d}s \ \mathrm{d}x \tag{65}$$

satisfies, for all $t \geq 0$

$$\begin{aligned} K_2'(t) \leq & [\delta - (1-\ell)] \|u_t(t)\|_2^2 + [\delta - (1-\ell)] \|v_t(t)\|_2^2 - \frac{g(0)C_*^2}{4\delta a_0}\left(g' \diamond \nabla\vartheta^t\right)(t) \\ & + \delta c a(u(t),u(t)) + \delta c a(v(t),v(t)) - \frac{g(0)C_*^2}{4\delta a_0}\left(g' \diamond \nabla\varphi^t\right)(t) \\ & + \left[\frac{2\ell(1-\ell)}{4\delta a_0} + \frac{1-\ell}{2a_0}(a_1+2) + \frac{(1-\ell)C_*^2}{4\delta a_0}\right] \times \\ & \left[\left(g \diamond \nabla\vartheta^t\right)(t) + \left(g \diamond \nabla\varphi^t\right)(t)\right], \end{aligned} \tag{66}$$

in which δ is some positive constant.

Proof. Using (20), we obtain

$$\begin{aligned} R_1'(t) = & -\int_\Omega u_{tt}(t)\int_0^\infty g(s)\vartheta^t(s)\mathrm{d}s \ \mathrm{d}x - \int_\Omega u_t(t)\int_0^\infty g'(s)\vartheta^t(s)\mathrm{d}s \ \mathrm{d}x \\ & -(1-\ell)\int_\Omega u_t^2(t)\mathrm{d}x \\ = & \ell \sum_{i,j=1}^{2}\int_0^\infty g(s)\int_\Omega a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial \vartheta^t(s)}{\partial x_i}dxds \\ & + \sum_{i,j=1}^{2}\int_\Omega\left(\int_0^\infty g(s)a_{ij}(x)\frac{\partial\vartheta^t(s)}{\partial x_j}ds\right)\left(\int_0^\infty g(s)\frac{\partial\vartheta^t(s)}{\partial x_i}ds\right)dx \\ & + \int_\Omega\left(\int_0^\infty g(s)\vartheta^t(s)ds\right)\kappa v^{2p}u^{2p-1}\ln\left(|uv|\right)dx \\ & - \int_\Omega u_t(t)\int_0^\infty g'(s)\vartheta^t(s)\mathrm{d}s \ \mathrm{d}x - (1-\ell)\int_\Omega u_t^2(t)\mathrm{d}x. \end{aligned} \tag{67}$$

By exploiting Young's inequality, we get that, for any $\delta > 0$,

$$\begin{aligned} & \left|-\ell\sum_{i,j=1}^{2}\int_0^\infty g(s)\int_\Omega a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial\vartheta^t(s)}{\partial x_i} \ \mathrm{d}s \ \mathrm{d}x\right| \\ & \leq \frac{\delta a_1 \ell}{a_0}a(u(t),u(t)) + \frac{2\ell(1-\ell)}{4\delta a_0}\left(g \diamond \nabla\vartheta^t\right)(t). \end{aligned} \tag{68}$$

The second term in the right-hand side of (67) can be estimated by

$$\begin{aligned} &\left| -\sum_{i,j=1}^{2} \int_\Omega \left(\int_0^\infty g(s) a_{ij}(x) \frac{\partial \vartheta^t(s)}{\partial x_j} \,\mathrm{d}s \right) \left(\int_0^\infty g(s) \frac{\partial \vartheta^t(s)}{\partial x_i} \,\mathrm{d}s \right) \mathrm{d}x \right| \\ &\le \frac{1}{2} \sum_{i,j=1}^{2} \int_\Omega \left| \int_0^\infty g(s) a_{ij}(x) \frac{\partial \vartheta^t(s)}{\partial x_j} \,\mathrm{d}s \right|^2 \mathrm{d}x \\ &+ \frac{1}{2} \sum_{i,j=1}^{2} \int_\Omega \left| \int_0^\infty g(s) \frac{\partial \vartheta^t(s)}{\partial x_i} \,\mathrm{d}s \right|^2 \mathrm{d}x \\ &\le \frac{1-\ell}{2a_0} [a_1 + 2] \left(g \diamond \nabla \vartheta^t \right)(t). \end{aligned} \tag{69}$$

Using Young's inequality, we get

$$|-\int_\Omega u_t(t) \int_0^\infty g'(s) \vartheta^t(s) \mathrm{d}s \;\mathrm{d}x| \le \delta \left\| u_t(t) \right\|_2^2 - \frac{g(0) C_*^2}{4\delta a_0} \left(g' \diamond \nabla \vartheta^t \right)(t). \tag{70}$$

Applying (22) for $y = |uv|^{2p-1}$,we get, for any $\delta > 0$,

$$\begin{aligned} &\left| \int_\Omega \left(\int_0^\infty g(s) \vartheta^t(s) ds \right) \kappa v^{2p} u^{2p-1} \ln \left(|uv| \right) dx \right| \\ &\le \frac{1}{2p-1} \int_\Omega \left| \left(\int_0^\infty g(s) \vartheta^t(s) ds \right) \right| |v| |vu|^{2p-1} \left| \ln \left(|uv|^{2p-1} \right) \right| dx \\ &\le \int_\Omega \left| \left(\int_0^\infty g(s) \vartheta^t(s) ds \right) \right| |v| \left(|uv|^{4p-2} + \tau \sqrt{|uv|^{2p-1}} \right) dx \\ &\le \delta \int_\Omega |v|^2 \left(|uv|^{4p-2} + \tau \sqrt{|uv|^{2p-1}} \right)^2 dx + \frac{(1-\ell) C_*^2}{4\delta a_0} \left(g \diamond \nabla \vartheta^t \right)(t) \\ &\le 2\delta \int_\Omega |v|^2 \left(|uv|^{8p-4} + \tau^2 |uv|^{2p-1} \right) dx + \frac{(1-\ell) C_*^2}{4\delta a_0} \left(g \diamond \nabla \vartheta^t \right)(t) \\ &\le c\delta a(u(t), u(t)) + c\delta a(v(t), v(t)) + \frac{(1-\ell) C_*^2}{4\delta a_0} \left(g \diamond \nabla \vartheta^t \right)(t). \end{aligned} \tag{71}$$

Combining all above inequalities, we get

$$\begin{aligned} R_1'(t) &\le \left[\delta - (1-\ell) \right] \left\| u_t(t) \right\|_2^2 - \frac{g(0) C_*^2}{4\delta a_0} \left(g' \diamond \nabla \vartheta^t \right)(t) + \delta c a(u(t), u(t)) \\ &+ \delta c a(v(t), v(t)) + \left[\frac{2\ell(1-\ell)}{4\delta a_0} + \frac{1-\ell}{2a_0}(a_1 + 2) + \frac{(1-\ell) C_*^2}{4\delta a_0} \right] \times \\ &\left(g \diamond \nabla \vartheta^t \right)(t). \end{aligned} \tag{72}$$

In the same manner, we obtain

$$\begin{aligned} R_2'(t) &\le \left[\delta - (1-\ell) \right] \left\| v_t(t) \right\|_2^2 - \frac{g(0) C_*^2}{4\delta a_0} \left(g' \diamond \nabla \varphi^t \right)(t) + \delta c a(v(t), v(t)) \\ &+ \delta c a(u(t), u(t)) + \left[\frac{2\ell(1-\ell)}{4\delta a_0} + \frac{1-\ell}{2a_0}(a_1 + 2) + \frac{(1-\ell) C_*^2}{4\delta a_0} \right] \times \\ &\left(g \diamond \nabla \varphi^t \right)(t). \end{aligned} \tag{73}$$

Together with (72) and (73), the proof of Lemma 8 is now completed.

□

Now, let us construct a Lyapunov functional L as

$$L(t) = ME(t) + N_1K_1(t) + N_2K_2(t) \tag{74}$$

where $M,\ N_1$ and N_2 are positive constants which will be choose later.

Lemma 9. *For $M > 0$ large enough, there exist two positive constants σ_1 and σ_2 such that*

$$\sigma_1 E(t) \leq L(t) \leq \sigma_2 E(t). \tag{75}$$

Proof. By Cauchy-Schwarz's, Young's and Poincaré's inequalities, we have

$$\begin{aligned} |K_1(t)| \leq & \Big|\int_\Omega u_t(t)u(t)\mathrm{d}x\Big| + \Big|\int_\Omega v_t(t)v(t)\mathrm{d}x\Big| \\ \leq & \frac{1}{2}\left\|u_t(t)\right\|_2^2 + \frac{C_*^2}{2a_0}a(u(t),u(t)) + \frac{1}{2}\left\|v_t(t)\right\|_2^2 + \frac{C_*^2}{2a_0}a(v(t),v(t)) \end{aligned} \tag{76}$$

and

$$\begin{aligned} |R_1(t)| &\leq \frac{1}{2}\left\|u_t(t)\right\|_2^2 + \frac{1}{2}\int_\Omega \left(\int_0^\infty g(s)\vartheta^t(s)\mathrm{d}s\right)^2 dx \\ &\leq \frac{1}{2}\left\|u_t(t)\right\|_2^2 + \frac{(1-l)C_*^2}{2a_0}\left(g \diamond \nabla\vartheta^t\right)(t). \end{aligned} \tag{77}$$

Similarly, we have

$$|R_2(t)| \leq \frac{1}{2}\left\|v_t(t)\right\|_2^2 + \frac{(1-l)C_*^2}{2a_0}\left(g \diamond \nabla\varphi^t\right)(t). \tag{78}$$

Combining (74), (76)-(78), we get

$$\begin{aligned} &|L(t) - ME(t)| \\ &\leq \frac{1}{2}\left(N_1+N_2\right)\left\|u_t(t)\right\|_2^2 + \frac{N_1C_*}{2a_0}a(u(t),u(t)) + \frac{N_2(1-\ell)C_*^2}{2a_0}\left(g \diamond \nabla\vartheta^t\right)(t) \\ &+ \frac{1}{2}\left(N_1+N_2\right)\left\|v_t(t)\right\|_2^2 + \frac{N_1C_*}{2a_0}a(v(t),v(t)) + \frac{N_2(1-\ell)C_*^2}{2a_0}\left(g \diamond \nabla\varphi^t\right)(t) \\ &\leq c\left(\frac{1}{2}\left(g \diamond \nabla\vartheta^t\right)(t) + \frac{1}{2}\left\|u_t(t)\right\|_2^2 + \frac{1}{2}\left\|v_t(t)\right\|_2^2 + \frac{\ell}{2}a(u(t),u(t)) \right. \\ &\left. + \frac{\ell}{2}a(v(t),v(t)) + \frac{1}{2}\left(g \diamond \nabla\varphi^t\right)(t) - \kappa\int_\Omega F(u,v)dx\right) + c\kappa\int_\Omega F(u,v)dx. \end{aligned} \tag{79}$$

Using (44) and (56), we obtain

$$|L(t) - ME(t)| \leq cE(t) + c\left[a(u(t),ut)) + a(v(t),v(t))\right]^{3p} \leq cE(t). \tag{80}$$

$\square$

Lemma 10. *The Lyapunov functional L defined in (74) satisfies, for all $t \geq 0$*

$$L'(t) \leq -mE(t) + c\left(g \diamond \nabla\vartheta^t\right)(t) + c\left(g \diamond \nabla\varphi^t\right)(t) \tag{81}$$

where m is a positive constant.

Proof. Combining (27), (58), (66) and taking $\delta = \frac{l}{cN_2}$, we have, for all $t \geq 0$,

$$\begin{aligned} L'(t) \leq &-\frac{\ell}{2}\left[N_1 - 2\right]\Big(a(u(t),u(t)) + a(v(t),v(t))\Big) \\ &- \left[N_2(1-\ell) - N_1 - \frac{\ell}{c}\right]\Big(\left\|u_t(t)\right\|_2^2 + \left\|v_t(t)\right\|_2^2\Big) \\ &+ \frac{1}{2}\left[M - cN_2^2\right]\left(\left(g' \diamond \nabla\varphi^t\right)(t) + \left(g' \diamond \nabla\vartheta^t\right)(t)\right) \\ &+ c\left(\left(g \diamond \nabla\vartheta^t\right)(t) + \left(g \diamond \nabla\varphi^t\right)(t)\right). \end{aligned} \tag{82}$$

At this point, we choose $N_1 > 0$ so that

$$N_1 - 2 > 4.$$

When N_1 is fixed, we pick $N_2 > 0$ so large that

$$N_2(1-\ell) - N_1 - \frac{\ell}{c} > 2.$$

Then we choose M large enough satisfying

$$M - cN_2^2 > 0.$$

So, we have

$$\begin{aligned} L'(t) \leq &- 2\ell\Big(a(u(t),u(t)) + a(v(t),v(t))\Big) - 2\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) \\ &+ c\left(\left(g \diamond \nabla\vartheta^t\right)(t) + \left(g \diamond \nabla\varphi^t\right)(t)\right) - \kappa\int_\Omega F(u(t),v(t))dx \\ &+ \kappa\int_\Omega F(u(t),v(t))dx. \end{aligned} \tag{83}$$

Therefore,

$$\begin{aligned} L'(t) \leq &- mE(t) - \ell\Big(a(u(t),u(t)) + a(v(t),v(t))\Big) + \kappa\int_\Omega F(u(t),v(t))dx \\ &+ c\left(\left(g \diamond \nabla\vartheta^t\right)(t) + \left(g \diamond \nabla\varphi^t\right)(t)\right). \end{aligned} \tag{84}$$

Using (43), (45) and (46), we get

$$\begin{aligned} &-\ell\Big(a(u(t),u(t)) + a(v(t),v(t))\Big) + \kappa\int_\Omega F(u(t),v(t))dx \\ &\leq \kappa\frac{(3E(0))^{3p-1}C_*^{6p}}{4p^2\ell^{3p-1}a_0^{3p}}\Big(a(u(t),u(t)) + a(v(t),v(t))\Big) - \ell a(u(t),u(t)) \\ &\quad - \ell a(v(t),v(t)) \\ &\leq \frac{\ell}{6}a(u(t),u(t)) + \frac{\ell}{6}a(v(t),v(t)) - \ell\left(a(u(t),u(t)) + a(v(t),v(t))\right) \leq 0. \end{aligned} \tag{85}$$

So,

$$L'(t) \leq - mE(t) + c\left(\left(g \diamond \nabla\vartheta^t\right)(t) + \left(g \diamond \nabla\varphi^t\right)(t)\right). \tag{86}$$

□

5 Stability result

Theorem 3. *Let (43) hold and (u, v) be the solution of (20). Then, there exist two positive constants q_1 and q_2 such that the energy of problem (20) satisfies*

$$E(t) \leq q_1 e^{-q_2\int_0^t \xi(s)ds}, \qquad \forall t \geq 0. \tag{87}$$

Proof. By multiplying (86) by ξ, recalling (A2) and using (27), we arrive at

$$\begin{aligned} \xi(t)L'(t) &\leq -m\xi(t)E(t) + c\xi(t)\left(\left(g \diamond \nabla\vartheta^t\right)(t) + \left(g \diamond \nabla\varphi^t\right)(t)\right) \\ &\leq -m\xi(t)E(t) - c\left(\left(g' \diamond \nabla\vartheta^t\right)(t) + \left(g' \diamond \nabla\varphi^t\right)(t)\right) \\ &\leq -m\xi(t)E(t) - cE'(t). \end{aligned} \tag{88}$$

That is

$$\big(\xi(t)L'(t) + cE(t)\big)' - \xi'(t)L(t) \leq - m\xi(t)E(t). \tag{89}$$

Using the fact that $\xi'(t) \leq 0, \forall t \geq 0$ and letting

$$\xi(t)L'(t) + cE(t) = F(t) \sim E(t) \tag{90}$$

we obtain

$$F'(t) \leq - m\xi(t)E(t) \leq -m_2\xi(t)F(t). \tag{91}$$

A simple integration of (91) over $(0, t)$ leads to

$$F(t) \leq F(t_0)e^{-m\int_0^t \xi(s)ds}. \tag{92}$$

The proof of Theorem 3 is thus completed. □

6 Conclusion

This paper deals with coupled wave system with viscoelastic terms and new logarithmic nonlinearities. It is well known that f_1 and f_2 in (8) have a relation (see for example Said-Houari et al. (2011); Han & Wang (2009); Mustafa (2012); Messaoudi & Al-Gharabli (2015)), which is the existence of a **positive** function $F(u, v)$ satisfying

$$\begin{cases} f_1(u,v) = \dfrac{\partial F}{\partial u}(u,v), \\ \\ f_2(u,v) = \dfrac{\partial F}{\partial v}(u,v), \\ \\ \mathbf{F(u,v) \geq 0}. \end{cases} \tag{93}$$

But in our system $f_1(u, v) = \frac{\partial F}{\partial u}(u, v)$ and $f_2(u, v) = \frac{\partial F}{\partial v}(u, v)$ where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by

$$F(u,v) = \frac{1}{4p^2}\left(\kappa v^{2p}u^{2p}\left(2p\ln\left(|uv|\right) - 1\right)\right),$$

is not necessarily a positive function. Our goal in this work was to eliminate the condition of positivity of the function $F(u, v)$ and give a new result concerning the existence and stability of the solution of nonlinear viscoelastic wave systems.

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References

Al-Gharabli, M.M., Guesmia, A., & Messaoudi, S.A. (2019). Existence and a general decay results for a viscoelastic plate equation with a logarithmic nonlinearity. *Commun. Pure Appl. Anal.*, *18*(1), 159–180.

Al-Gharabli, M.M., Guesmia, A., & Messaoudi, S.A. (2020). Well-posedness and asymptotic stability results for a viscoelastic plate equation with a logarithmic nonlinearity. *Appl. Anal.*, *99*(1), 50–74.

Barrow, J.D., Parsons, P. (1995). Inflationary models with logarithmic potentials. *Phys. Rev. D.*, *52*(10), 5576.

Bartkowski, K., Gorka, P. (2008). One-dimensional klein-gordon equation with logarithmic nonlinearities. *J. Phys. A, Math. Theor.*, *41*(35), 11.

Białynicki-Birula, I., Mycielski, J. (1975). Wave equations with logarithmic nonlinearities. *Bull. Acad. Pol. Sci. Cl.*, *3*(23), 461–466.

Boukhatem, Y., Benabderrahmane, B. (2014). Existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions. *Nonlinear Anal.*, *97*, 191–209.

Boukhatem, Y., Benabderrahmane, B. (2020). Asymptotic behavior for a past history viscoelastic problem with acoustic boundary conditions. *Appl. Anal.*, *99*(2), 249–269.

Boulaaras, S. (2021). A well-posedness and exponential decay of solutions for a coupled Lamé system with viscoelastic term and logarithmic source terms. *Appl. Anal.*, *100*(7), 1514–1532.

Cazenave, T., Haraux, A. (1980). Equations d'évolution avec non linearite logarithmique. *Ann. Fac. Sci. Toulouse, Math.,* *5*(2), 21–51.

Chellaoua, H., Boukhatem, Y., & Feng, B. (2022). Optimal decay of an abstract nonlinear viscoelastic equation in Hilbert spaces with delay term in the nonlinear internal damping. *Asymptotic. Anal.*, *126*(1-2), 65–94.

Dafermos, C. (1970). Asymptotic stability in viscoelasticity. *Arch Rational Mech Anal.*, *37*(4), 297–308.

Enqvist, K., McDonald, J. (1998). Q-balls and baryogenesis in the mssm. *Phys. Lett. B*, *425*(3-4), 309–321.

Górka, P. (2009). Logarithmic Klein-Gordon equation. *Acta Phys. Pol. B*, *40*(1), 59–66.

Guesmia, A. (2011). Asymptotic stability of abstract dissipative systems with infinite memory. *J. Math. Anal. Appl.*, *382*(2), 748–760.

Han, X. (2013). Global existence of weak solutions for a logarithmic wave equation arising from q-ball dynamics. *Bull. Korean Math. Soc.*, *50*(1), 275–283.

Han, X., Wang, M. (2009). Global existence and blow-up of solutions for a system of nonlinear viscoelastic wave equations with damping and source. *Nonlinear Anal.*, *71*(11), 5427–5450.

Hiramatsu, T., Kawasaki, M., & Takahashi, F. (2010). Numerical study of q-ball formation in gravity mediation. *J. Cosmol. Astropart. Phys.*, *2010*(6), 008.

Hu, Q., Zhang, H., & Liu, G. (2019). Asymptotic behavior for a class of logarithmic wave equations with linear damping. *Appl. Math. Optim.*, *79*(1), 131–144.

Irkıl, N., Piskin, E. (2021). On decay and blow-up for a system of viscoelastic wave equations with logarithmic nonlinearity. *Advanced Studies: Euro-Tbilisi Mathematical Journal*, *10*, 65–82.

Irkıl, N., Pişkin, E., & Agarwal, P. (2022). Global existence and decay of solutions for a system of viscoelastic wave equations of kirchhoff type with logarithmic nonlinearity. *Math. Methods Appl. Sci.*, *45*(5), 2921–2948.

Kafini, M., Messaoudi, S. (2020). Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay. *Appl. Anal.*, *99*(3), 530–547.

Komornik, V. (1994). *Exact controllability and stabilization. The multiplier method.* Chichester: Wiley, Masson, Paris.

Limam, A., Boukhatem, Y., & Benabderrahmane, B. (2021). New general stability for a variable coefficient thermo-viscoelastic-coupled system of second sound with acoustic boundary conditions. *Comput. Appl. Math.*, *40*(3), 20.

Messaoudi, S.A., Al-Gharabli, M.M. (2015). A general decay result of a nonlinear system of wave equations with infinite memories. *Appl. Math. Comput.*, *259*, 540–551.

Mustafa, M.I. (2012). Well posedness and asymptotic behavior of a coupled system of nonlinear viscoelastic equations. *Nonlinear Anal. Real World Appl.*, *13*(1), 452–463.

Pazy, A. (1983). *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Appl. Math. Sci.*, Springer, Cham, New York.

Peyravi, A. (2020). General stability and exponential growth for a class of semi-linear wave equations with logarithmic source and memory terms. *Appl. Math. Optim.*, *81*(2), 545–561.

Said-Houari, B., Messaoudi, S.A., & Guesmia, A. (2011). General decay of solutions of a nonlinear system of viscoelastic wave equations. *NoDEA. Nonlinear Differ. Equ. Appl.*, *18*(6), 659–684.

Wang, X., Chen, Y., Yang, Y., Li, J., & Xu, R. (2019). Kirchhoff-type system with linear weak damping and logarithmic nonlinearities. *Nonlinear Anal.*, *188*, 475–499.